On a rod motion in a rarefied particle medium

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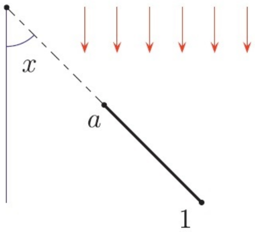
Flight of rod

Rod in plane rarefied flow

Team work with Alexander Plakhov.



$2(v\sin x + r\dot{x})$ – velocity change of particle reflected at $r \in [a, 1]$;
$\rho|v\sin x + r\dot{x}|drdt$ – mass of particles reflected from a segment $[r, r+dr]$ during dt;
$2\rho\int_a^1 (v\sin x + r\dot{x})|v\sin x + r\dot{x}|\, r\, dr$ – moment of force acting on the rod.
Equation of pendulum dynamic is

$$I\ddot{x} = -2\rho\int_a^1 (v\sin x + r\dot{x})|v\sin x + r\dot{x}|\, r\, dr.$$

Four regions of different rod dynamics: equations

Make the change of variable $\tau = vt$ and introduce k, $k = 2\rho/I$:

$$\ddot{x} = k \int_a^1 (\sin x + r\dot{x}) |\sin x + r\dot{x}|\, r\, dr.$$

(a) $\sin x + r\dot{x} \geq 0$ for all $r \in [a, 1]$ or $\dot{x} \geq \max\left\{-\sin x, -\dfrac{\sin x}{a}\right\}$;

$$\ddot{x} = -k\left[\frac{1-a^2}{2}\sin^2 x + 2\,\frac{1-a^3}{3}\,\dot{x}\sin x + \frac{1-a^4}{4}\,\dot{x}^2\right].$$

(b) $\sin x + a\dot{x} \geq 0$ and $\sin x + \dot{x} < 0$, or $-\dfrac{\sin x}{a} \leq \dot{x} < -\sin x$:

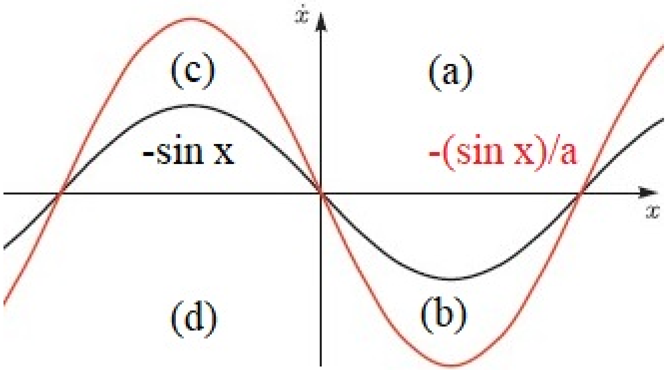
$$\ddot{x} = k\left\{\left[\frac{1+a^2}{2}\sin^2 x + 2\,\frac{1+a^3}{3}\,\dot{x}\sin x + \frac{1+a^4}{4}\,\dot{x}^2\right] - \frac{\sin^4 x}{6\dot{x}^2}\right\}.$$

(c) $\sin x + a\dot{x} < 0$ and $\sin x + \dot{x} \geq 0$, or $-\sin x \leq \dot{x} < -\dfrac{\sin x}{a}$.

(d) $\sin x + r\dot{x} < 0$ for all $r \in [a, 1]$, or $\dot{x} < \min\left\{-\sin x, -\dfrac{\sin x}{a}\right\}$:

(c) $\ddot{x}$ = - case(b) (d) $\ddot{x}$ = - case(b)

Four regions of different rod dynamics



Singular points I

With $y := \dot{x}$ we have

$$\dot{x} = y \quad \text{and} \quad \dot{y} = v(x, y)$$

where $x(x, y)$ is the respective expression in the domains (a)-(d). Singular points are $(0, 0)$, $(\pi, 0)$ up to x-period 2π. For the part of (d) with $\pi \leq x \leq \pi + \epsilon$ introduce $z > 0, w$ with $x = \pi + z^2$ and $y = wz^3$. That gives $\dot{z} = wz^2/2$ and

$$\dot{w} = k\left\{-\frac{3}{2}\,w^2 z + \frac{1-a^2}{2}\,\frac{\sin^2 z^2}{z^3} + 2\,\frac{1-a^3}{3} w \sin z^2 + \frac{1-a^4}{4} w^2 z^3\right\}.$$

After the division the vector field by $z > 0$ we get $\dot{z} = wz/2$ and

$$\dot{w} = k\left\{-\frac{3}{2}\,w^2 + \frac{1-a^2}{2}\frac{\sin^2 z^2}{z^4} + 2\,\frac{1-a^3}{3}\frac{\sin z^2}{z^2} wz + \frac{1-a^4}{4} w^2 z^2\right\}$$

Singular points II

$$z_0 = 0, \quad w_0 = w_\pm \text{ with } w_\pm = \pm\sqrt{\frac{1-a^2}{3}}.$$

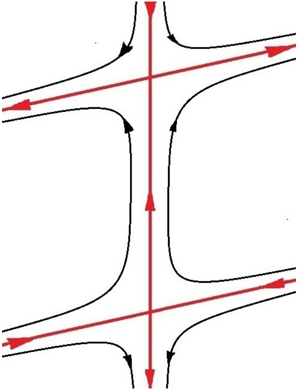
The matrix of linearization of the field at these points is

$$\begin{pmatrix} w_0/2 & 0 \\ 2\frac{1-a^3}{3}kw_0 & -3kw_0 \end{pmatrix}. \tag{1}$$

Its eigenvalues $w_0/2$ and $-3kw_0$ are real and have different signs, hence the points are non-degenerate saddles. The corresponding eigendirections are $(18k+3,\ 4k(1-a^3))$ and $(0,1)$.

Singular point $(\pi, 0)$ I

In the whole the phase portrait near the w-axis looks like



Singular points $(\pi, 0)$ II

Consider domains (b) and (c) near the point. Let it be (c).

$$\dot{x} = y,\ \dot{y} = -k\left[\frac{1+a^2}{2}\sin^2 x + 2\,\frac{1+a^3}{3}\,y\sin x + \frac{1+a^4}{4}\,y^2\right] + k\frac{\sin^4 x}{6y^2}.$$

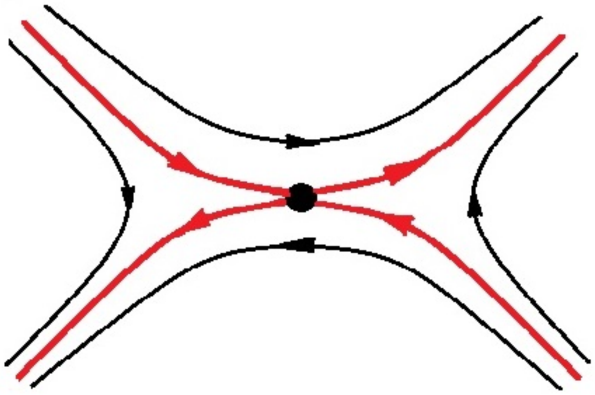
With $w, y = w\sin x$, $-1/a \le w \le -1$, and after the division of the second equation by $\sin x$ we get $\dot{x} = w\sin x$ and

$$\dot{w} = -w^2\cos x - k\left[\frac{1+a^2}{2} + 2\,\frac{1+a^3}{3}w + \frac{1+a^4}{4}w^2\right]\sin x + k\frac{\sin x}{6w^2}.$$

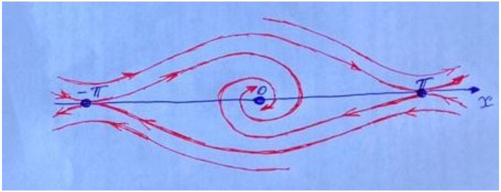
The last field is smooth near segment $\{x = \pi,\ -1/a \le w \le -1\}$. Hence for x sufficiently close to π the phase curves near the segment go up almost vertically

Singular points $(\pi, 0)$ III

Hence the singular point $(\pi, 0)$ is topologically saddle:



The full picture



Topologically it is the same like in V.I.Arnold's textbook.

$$\ddot{x} + a\dot{x} + kx = 0,\ a > 0,\ k > 0$$

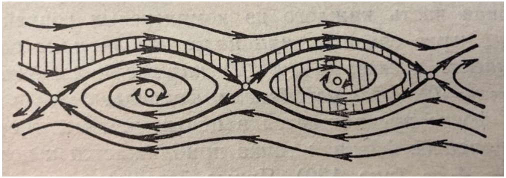
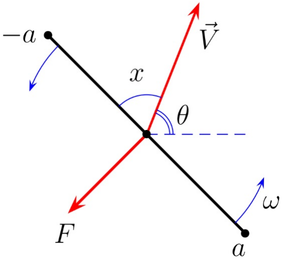


Figure: V.I.Arnold, Ordinary differential equations (various editions).

Flight of a rod



$$\begin{cases} \dot{v} = -\frac{2\rho}{m} \sin x \int_{-a}^{a} (r\omega + v \sin x)\, |r\omega + v\sin x|\, dr \\ \dot{x} = \omega - \frac{2\rho}{m} \frac{\cos x}{v} \int_{-a}^{a} (r\omega + v \sin x)\, |r\omega + v\sin x|\, dr \\ \dot{\omega} = -\frac{2\rho}{I} \int_{-a}^{a} (r\omega + v \sin x)\, |r\omega + v\sin x|\, r\, dr \end{cases}$$

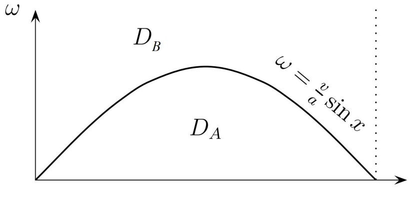
Calculations are different in 3 cases when the sign of $r\omega + v\sin x$:
A) is either non-negative ($=A_+$) or non-positive ($=A_-$) for all $r \in (-a, a)$,
B) changes sign at $r_0 = -v \sin x/\omega \in (-a, a)$, from "-" to "+" ($=B_-$), or vice versa.

Model equations

$$(A_+) \quad \begin{cases} \dot{v} &= -\frac{2\rho}{m} \sin x\left(\frac{2}{3}a^3\omega^2 + 2av^2\sin^2 x\right) \\ \dot{x} &= \omega - \frac{2\rho}{m} \frac{\cos x}{v}\left(\frac{2}{3}a^3\omega^2 + 2av^2\sin^2 x\right) \\ \dot{\omega} &= -\frac{8\rho a^3}{3I} v\omega\sin x. \end{cases} ,$$

$$(B_-) \quad \begin{cases} \dot{v} &= -\frac{4v\rho}{m}\sin^2 x\left(\frac{v^2\sin^2 x}{3\omega} + a^2\omega\right) \\ \dot{x} &= \omega - \frac{4\rho}{m} \cos x \sin x\left(\frac{v^2\sin^2 x}{3\omega} + a^2\omega\right) \\ \dot{\omega} &= -\frac{2\rho}{I}\left[\frac{v^2\sin^2 x}{\omega^2}\left(-\frac{1}{6}v^2\sin^2 x + a^2\omega^2\right) + \frac{a^4\omega^2}{2}\right] \end{cases}$$

Domain of study: $D := \{0 \le x \le \pi,\ 0 < v,\ 0 \le \omega\} = D_{A_+} \cup D_{B_-}$.
Boundary: $a\omega = v\sin x$, $0 \le x \le \pi$, $0 \le v$.



Stabilization I

Theorem

In the interior of the domain D, when moving, both ω and v decrease. In particular, the function $v + \omega$ decreases, and any trajectory of the model tends to the boundary of this domain.

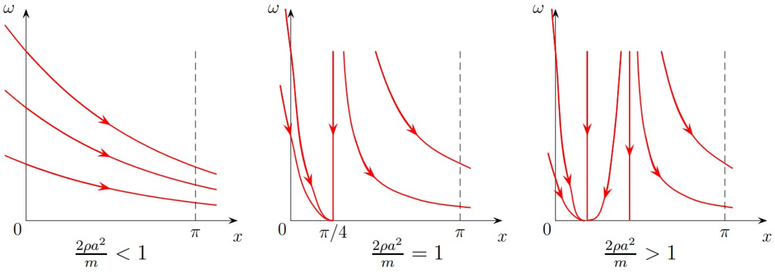
The boundary of domain D is the union of its five parts:

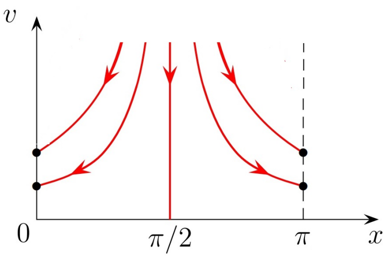
$$\textit{left} \ = \{x = 0,\ w > 0,\ v > 0\},\ \textit{right} \ = \{x = \pi,\ w > 0,\ v > 0\},$$

$$\textit{bottom} = \{x \in [0, \pi],\ w = 0,\ v > 0\},\ \textit{front} \ = \{x \in [0, \pi],\ w > 0,\ v = 0\}$$

$$\textit{corner} = \{x \in [0, \pi],\ w = v = 0\}.$$

Stabilization II





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Thanks for the attention!