An Axiomatic Characterization of Weighted Congestion Games

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Key notations and definitions

A *normal-form game* with identical players' strategy sets is a three-tuple $\Gamma = (N, S, \{u_i\}_{i \in N})$.

A *strategy profile* $s \in S^n$ is a vector $s = (s_1, s_2, ..., s_n)$, where $s_i \in S$ is the *strategy* of the player $i \in N$.

Seeking to isolate the strategy s_i of the player i in the profile $s \in S^n$, we write $s = (s_i, s_{-i})$, where $s_{-i} \in S^{n-1}$. Similarly, the strategies of the different players i and k are isolated by writing $s = (s_i, s_k, s_{-ik})$, where $s_i, s_k \in S, s_{-ik} \in S^{n-2}$.

Key notations and definitions

The *weighted congestion model* is a tuple

$$(N, M, S, \{w_i\}_{i \in N}, \{c_{ij}\}_{ij \in N \times M}),$$

where:

1) N and S have the same meaning as in the game Γ;
2) M is a finite set of resources;
3) the number $w_i > 0$ is the weight of the player i;
4) the resource function $c_{ij} : W \to \mathbb{R}^+$ is the payout function for the player i from the resource $j \in M$, where $W = \{\sum_{r \in K} w_r\}_{\substack{K \subseteq N \\ K \neq \emptyset}}$ is the set of all possible different sums of players' weights.

Key notations and definitions

The weighted congestion game (WCG) is a normal-form game $(N, S, \{u_i\}_{i \in N})$ in which the players' payoff functions have the form

$$u_i(s) = \sum_{j \in s_i} c_{ij} \left(\sum_{r \in K_j(s)} w_r \right),$$

where $K_j(s)$ is the set of players who have chosen the resource $j \in M$ in the profile $s \in S^n$.

Key notations and definitions

The weighted congestion game with players' constants (WCGC) is a WCG in which $c_{ij}(w) = \alpha_i \cdot c_j(w) \; \forall (i,j) \in N \times M \; \forall w \in W$, where $\alpha_i > 0$ is the constant for the player i and $c_j : W \to \mathbb{R}^+$. The payoff functions for players in the WCGC have the form

$$u_i(s) = \alpha_i \cdot \sum_{j \in s_i} c_j \left(\sum_{r \in K_j(s)} w_r \right).$$

Key notations and definitions

The weighted congestion game with player-independent resource functions (WCGI) is a WCGC for which $\alpha_i = 1 \; \forall i \in N$, that is the players' payoff functions have the form

$$u_i(s) = \sum_{j \in s_i} c_j \left(\sum_{r \in K_j(s)} w_r \right).$$

Key notations and definitions

Suppose the strategy set S somehow depends on M. We use $G(N, M, S)$ to denote a set consisting of all possible normal-form games $(N, S, \{u_i\}_{i \in N})$.

The cases of interest are where $S = M$ or $S = 2^M$. In the former and the latter cases the normal-form game is called a *singleton* or a *full* game, respectively.

Motivation questions

Axiomatization of weighted congestion games is motivated by the following questions:

1. Modeling of game-theoretic processes. Suppose some game-theoretic process possesses, for example, the Independence property. Weighted congestion games are known to possess this property. Is the Independence property alone sufficient for the game-theoretic process to be modeled by a weighted congestion game?
2. Choosing the congestion model. How does one decide which game to use as a model for the game-theoretic process?
3. Transfer to a model with independent parameters. Can we represent a game with dependent parameters as a weighted congestion game with independent parameters?

Positivity (P). Let $\Gamma \in G(N, M, S)$. For any strategy profile $s \in S^n$ and for any player $i \in N$ we have

$$u_i(s) > 0.$$

Independence of Irrelevant Choices (IIC). Let $\Gamma \in G(N, M, S)$. For any two different players $i, k \in N$, any profile $s = (s_i, s_k, s_{-ik}) \in S^n$, any strategy $s'_k \in S$ such that $s_i \cap s_k = \emptyset, s_i \cap s'_k = \emptyset$ we have

$$u_i(s_i, s_k, s_{-ik}) = u_i(s_i, s'_k, s_{-ik}).$$

A characterization of singleton WCGs

Theorem 1. *Let* $|N| \geq 2, |M| \geq 3, S = M$. *The game* $\Gamma \in G(N, M, S)$ *satisfies (P) and (IIC) axioms if and only if there exists an array of positive players' weights* $\{w_i\}_{i \in N}$ *and an array of resource functions* $\{c_{ij}\}_{i \in N, j \in M}, c_{ij} : W \to \mathbb{R}^+ \ \forall (i, j) \in N \times M$ *such that in the game* Γ *for* $\forall i \in N \ \forall s \in S^n$ *we have*

$$u_i(s) = c_{is_i}\left(\sum_{r \in K_{s_i}(s)} w_r\right).$$

Proof of Theorem 1

It is not hard to check that a game with the specified players' payoff functions satisfies (P) and (IIC) axioms.

Let the game Γ satisfy (P) and (IIC). Let us break the proof down into 3 points.

1. *Definition of weights and resource functions.* Let us set the positive numbers $w_1, w_2, ..., w_n$ so that

$$\sum_{r \in K_1} w_r \neq \sum_{r \in K_2} w_r \ \forall K_1, K_2 \subseteq N, K_1 \neq K_2.$$

Let $K \subseteq N, K \neq \emptyset, j \in M$. Denote

$$\overline{S}(K, j) = \{(s_1, s_2, ..., s_n) | s_i = j \ \forall i \in K \text{ and } s_i \neq j \ \forall i \in N \setminus K\}.$$

Since $|M| \geq 3$, the set $\overline{S}(K, j)$ consists of at least two strategy profiles.

Proof of Theorem 1

Let us for each player $i \in N$, for each resource $j \in M$ and for each coalition $K \subseteq N, i \in K$ set the value of the resource function c_{ij} as follows,

$$c_{ij}\left(\sum_{r \in K} w_r\right) = u_i(\overline{s}(K,j)) \ \forall \overline{s}(K,j) \in \overline{S}(K,j).$$

It follows from (P) that the values of players' payoff functions are positive. Hence, the resource functions also take positive values.

Proof of Theorem 1

2. *Uniqueness of resource functions.* Let us demonstrate that the function c_{ij} is defined uniquely for each pair $(i, j) \in N \times M$. Since all components of the vector $\left(\sum_{r \in K} w_r\right)_{\substack{K \subseteq N \\ K \neq \emptyset}}$ are pairwise different, we do not get a constraint on the resource function c_{ij} for the different coalitions K_1 and K_2, that is the numbers $c_{ij}\left(\sum_{r \in K_1} w_r\right)$ and $c_{ij}\left(\sum_{r \in K_2} w_r\right)$ for $K_1 \neq K_2$ can be either equal or not equal to each other.

Proof of Theorem 1

For the function c_{ij} to be defined uniquely it is sufficient that for each coalition $K \subseteq N, K \neq \emptyset$ and for each $i \in K$ the following equality is satisfied:

$$u_i(\overline{s}(K,j)) = u_i(\overline{s}'(K,j)) \ \ \forall \overline{s}(K,j), \overline{s}'(K,j) \in \overline{S}(K,j).$$

Such an equality follows from (IIC). Indeed, according to (IIC) axiom, we can replace the strategies of players from the coalition $N \setminus K$ in the profile $\overline{s}'(K,j)$ with corresponding players' strategies in the profile $\overline{s}(K,j)$ without inducing a change in the payoff of each player in the coalition K. Hence, the resource function c_{ij} for each pair $(i,j) \in N \times M$ is defined uniquely.

Proof of Theorem 1

3. *Representation of players' payoffs.* We have defined the values of players' weights and the values of resource functions. We can represent the payoff $u_i(s)$ as $u_i(\overline{s}(K_{s_i}(s), s_i))$, where $s = \overline{s}(K_{s_i}(s), s_i) \in \overline{S}(K_{s_i}(s), s_i)$. Then, for $\forall i \in N \; \forall s \in S^n$ we have

$$u_i(s) = u_i(\overline{s}(K_{s_i}(s), s_i)) = c_{is_i}\left(\sum_{r \in K_{s_i}(s)} w_r\right),$$

Q.E.D.

Resource Additivity (RA). Let $\Gamma \in G(N, M, S)$. For any strategy profiles $s = (s_k, s_{-k}) \in S^n, s' = (s'_k, s'_{-k}) \in S^n, s'' = (s''_k, s''_{-k}) \in S^n$, such that $s_k = s'_k \cup s''_k \; \forall k \in N$ and $(\cup_{k=1}^n s'_k) \cap (\cup_{k=1}^n s''_k) = \emptyset$, the players' payoff functions in the game Γ satisfy the equality

$$u_i(s) = u_i(s') + u_i(s'') \quad \forall i \in N.$$

A characterization of full WCGs

For example, let $N = \{1, 2, 3\}, M = \{a, b, c, d, e\}, S = 2^M$. Let us consider a game $\Gamma \in G(N, M, S)$ for which (RA) is satisfied. Suppose $s = (\{a, b\}, \{b, c, d\}, \{d, e\}) \in S^n$. In that case, we can decompose the profile s, for example, into the profiles $s' = (\{a, b\}, \{b, c\}, \emptyset)$ and $s'' = (\emptyset, \{d\}, \{d, e\})$. Hence, players' payoff functions in the game Γ satisfy the equality

$$u_i(\{a, b\}, \{b, c, d\}, \{d, e\}) = u_i(\{a, b\}, \{b, c\}, \emptyset) + u_i(\emptyset, \{d\}, \{d, e\}) \ \forall i \in N.$$

Continuing the decomposition process, we can get the following equality

$$u_i(\{a, b\}, \{b, c, d\}, \{d, e\}) = u_i(\{a\}, \emptyset, \emptyset) + u_i(\{b\}, \{b\}, \emptyset)$$

$$+u_i(\emptyset, \{c\}, \emptyset) + u_i(\emptyset, \{d\}, \{d\}) + u_i(\emptyset, \emptyset, \{e\}) \ \forall i \in N.$$

A characterization of full WCGs

Non-Negativity (NN). Let $\emptyset \in S$ and $\Gamma \in G(N, M, S)$. For any player $i \in N$ and for any profile $s = (s_i, s_{-i}) \in S^n$ we have

$$u_i(s_i, s_{-i}) > 0 \Leftrightarrow s_i \neq \emptyset \text{ and } u_i(s_i, s_{-i}) = 0 \Leftrightarrow s_i = \emptyset.$$

Theorem 2. *Let $|N| \geq 2, |M| \geq 2, S = 2^M$. The game $\Gamma \in G(N, M, S)$ satisfies (NN) and (RA) axioms if and only if there exists an array of positive players' weights $\{w_i\}_{i \in N}$ and an array of resource functions $\{c_{ij}\}_{i \in N, j \in M}, c_{ij} : W \to \mathbb{R}^+ \ \forall (i, j) \in N \times M$ such that in the game Γ for $\forall i \in N \ \forall s \in S^n$ we have*

$$u_i(s) = \sum_{j \in s_i} c_{ij} \left(\sum_{r \in K_j(s)} w_r \right)$$

when $s_i \neq \emptyset$ and $u_i(s) = 0$ when $s_i = \emptyset$.

A characterization of full WCGs

Lemma 1. *Let* $S = 2^M$ *and* $\Gamma \in G(N, M, S)$. *Then, the following two statements are true:*

1. The game Γ *satisfies (RA) if and only if for* $\forall i \in N \; \forall s \in S^n$ *we have*

$$u_i(s) = \sum_{j \in \bigcup_{k \in N} s_k} u_i(s_1 \cap \{j\}, s_2 \cap \{j\}, \ldots, s_n \cap \{j\}).$$

2. If the game Γ *satisfies (NN) and (RA), then for* $\forall i \in N \; \forall s \in S^n$ *we have*

$$u_i(s) = \sum_{j \in s_i} u_i(s_1 \cap \{j\}, s_2 \cap \{j\}, \ldots, s_n \cap \{j\})$$

when $s_i \neq \emptyset$ *and* $u_i(s) = 0$ *when* $s_i = \emptyset$.

A characterization of full WCGs without (RA)

Transfer (T). Let $\Gamma \in G(N, M, S)$. Then, for any two players $i, k \in N$ and for any profiles $(s_k, s_{-k}), (s'_k, s_{-k}) \in S^n$ such that $s_k \cup s'_k, s_k \cap s'_k \in S$ we have

$$u_i(s_k, s_{-k}) + u_i(s'_k, s_{-k}) = u_i(s_k \cup s'_k, s_{-k}) + u_i(s_k \cap s'_k, s_{-k}).$$

Theorem 3. *Let* $|N| \geq 2, |M| \geq 2, S = 2^M$. *The game* $\Gamma \in G(N, M, S)$ *satisfies (NN), (T) and (IIC) axioms if and only if there exists an array of positive players' weights* $\{w_i\}_{i \in N}$ *and an array of resource functions* $\{c_{ij}\}_{i \in N, j \in M}, c_{ij} : W \to \mathbb{R}^+ \ \forall (i,j) \in N \times M$ *such that in the game* Γ *for* $\forall i \in N \ \forall s \in S^n$ *we have*

$$u_i(s) = \sum_{j \in s_i} c_{ij} \left(\sum_{r \in K_j(s)} w_r \right)$$

when $s_i \neq \emptyset$ *and* $u_i(s) = 0$ *when* $s_i = \emptyset$.

A characterization of full WCGs without (RA)

Lemma 2. *Let $S = 2^M, \Gamma \in G(N, M, S)$ and Γ satisfies (NN) and (T) axioms. Then for $\forall i \in N\ \forall s \in S^n$ we have*

$$u_i(s) = \sum_{j \in s_i} u_i(\{j\}, s_{-i})$$

when $s_i \neq \emptyset$ and $u_i(s) = 0$ when $s_i = \emptyset$.

Lemma 3. *Let $S = 2^M, \Gamma \in G(N, M, S)$ and Γ satisfies (T) and (IIC) axioms. Then for any different players $i, k \in N$ and any profile $(\{j\}, s_k, s_{-ik}) \in S^n$, such that $j \in s_k$ we have*

$$u_i(\{j\}, s_k, s_{-ik}) = u_i(\{j\}, \{j\}, s_{-ik}).$$

Lemma 4. *Let $S = 2^M$. Then we have the following property of the axioms:*

$$(NN), (T), (IIC) \Rightarrow (RA).$$

A characterization of WCGs

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<tr><th>The game</th><th>The set of strategies</th><th>Conditions for $|M|, |N|$</th><th>Axioms</th></tr>
<tr><td rowspan="4">WCG</td><td rowspan="2">M</td><td>$|N| = 1$ or $1 \leq |M| \leq 2$</td><td>(P)</td></tr>
<tr><td>$|N| \geq 2, |M| \geq 3$</td><td>(P), (IIC)</td></tr>
<tr><td rowspan="3">2^M</td><td>$|N| = 1$ or $|M| = 1$</td><td>(NN)</td></tr>
<tr><td rowspan="2">$|N| \geq 2, |M| \geq 2$</td><td>(NN), (RA)</td></tr>
<tr><td></td><td>(NN), (T), (IIC)</td></tr>
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Symmetry (S). Let $\Gamma \in G(N, M, S)$. For any two different players $i, k \in N$ and any profile $s = (s_i, s_k, s_{-ik}) \in S^n$ such that $s_i = s_k$ the payoff functions for the players i and k in the game Γ satisfy the equality

$$u_i(s) = u_k(s).$$

Resource Marginal Contribution (RMC). Let $\Gamma \in G(N, M, S)$. For any two different players $i, k \in N$, and for any resource $j \in M$, for any profile $s = (s_i, s_k, s_{-ik}) \in S^n$ such that $j \in s_i, j \in s_k, s_i \setminus \{j\} \in S, s_k \setminus \{j\} \in S$ we have

$$u_i(s_i, s_k, s_{-ik}) - u_i(s_i \setminus \{j\}, s_k, s_{-ik}) = u_k(s_i, s_k, s_{-ik}) - u_k(s_i, s_k \setminus \{j\}, s_{-ik}).$$

A characterization of WCGIs

Lemma 5. *Let* $\Gamma \in G(N, M, S)$*. The following statements are true:*
1. Let $\emptyset \in S, \{j\} \in S$ *and* Γ *satisfies (NN) and (RMC) axioms. Then for any profile* $s \in S^n$*, such that* $s_i \in \{\emptyset, \{j\}\}, s_k = s_i$ *we have*

$$u_i(s) = u_k(s).$$

2. Let $S = M \cup \{\emptyset\}$*. Then we have the following property of the axioms:*

$$(NN), (RMC) \Rightarrow (S).$$

Theorem 4. *Let $|N| \geq 2, |M| \geq 3, S = M$. The game $\Gamma \in G(N, M, S)$ satisfies (P), (S) and (IIC) axioms if and only if there exists an array of positive players' weights $\{w_i\}_{i \in N}$ and an array of resource functions $\{c_j\}_{j \in M}, c_j : W \to \mathbb{R}^+ \ \forall j \in M$, such that in the game Γ for $\forall i \in N$ $\forall s \in S^n$ we have*

$$u_i(s) = c_{s_i}\left(\sum_{r \in K_{s_i}(s)} w_r\right).$$

A characterization of WCGIs

Theorem 5. *Let* $|N| \geq 2, |M| \geq 3, S = M \cup \{\emptyset\}$. *The game* $\Gamma \in G(N, M, S)$ *satisfies (P), (RMC) and (IIC) axioms if and only if there exists an array of positive players' weights* $\{w_i\}_{i \in N}$ *and an array of resource functions* $\{c_j\}_{j \in M}, c_j : W \to \mathbb{R}^+ \ \forall j \in M$, *such that in the game* Γ *for* $\forall i \in N \ \forall s \in S^n$ *we have*

$$u_i(s) = c_{s_i}\left(\sum_{r \in K_{s_i}(s)} w_r\right).$$

when $s_i \neq \emptyset$ *and* $u_i(s) = 0$ *when* $s_i = \emptyset$.

A characterization of WCGIs

Theorem 6. *Let* $|N| \geq 2, |M| \geq 2, S = 2^M$. *The game* $\Gamma \in G(N, M, S)$ *satisfies (NN), (RA) and (S) axioms if and only if there exists an array of positive players' weights* $\{w_i\}_{i \in N}$ *and an array of resource functions* $\{c_j\}_{j \in M}, c_j : W \to \mathbb{R}^+ \ \forall j \in M$, *such that in the game* Γ *for* $\forall i \in N$ $\forall s \in S^n$ *we have*

$$u_i(s) = \sum_{j \in s_i} c_j \left(\sum_{r \in K_j(s)} w_r \right)$$

when $s_i \neq \emptyset$ *and* $u_i(s) = 0$ *when* $s_i = \emptyset$.

A characterization of WCGIs

The game	The set of strategies	Conditions for $\lvert M\rvert, \lvert N\rvert$	Axioms
WCGI	M	$\lvert N\rvert = 1, \lvert M\rvert \geq 1$	(P)
		$\lvert N\rvert \geq 2, 1 \leq \lvert M\rvert \leq 2$	(P), (S)
		$\lvert N\rvert \geq 2, \lvert M\rvert \geq 3$	(P), (S), (IIC)
	2^M	$\lvert N\rvert = 1, \lvert M\rvert = 1$	(NN)
		$\lvert N\rvert = 1, \lvert M\rvert \geq 2$	(NN), (RA)
		$\lvert N\rvert \geq 2, \lvert M\rvert = 1$	(NN), (S)
		$\lvert N\rvert \geq 2, \lvert M\rvert \geq 2$	(NN), (RA), (S)
			(NN), (RA), (RMC)
			(NN), (T), (S), (IIC)
			(NN), (T), (RMC), (IIC)

A characterization of WCGCs

Proportional symmetry with maximal congestion (PS). Let $\Gamma \in G(N, M, S)$. For any different players $i, k \in N$, for any strategy $j \in S$ such that $u_l(j, j, ..., j) \neq 0 \; \forall l \in N$ and any strategy profile $s = (j, j, s_{-ik}) \in S^n$ we have

$$\frac{u_i(j, j, s_{-ik})}{u_i(j, j, ..., j)} = \frac{u_k(j, j, s_{-ik})}{u_k(j, j, ..., j)}.$$

Proportion of the maximal-congestion profile (PM). For any two different players $i, k \in N$ and for any two maximal-congestion profiles $s = (j, j, ..., j), s' = (j', j', ..., j') \in S^n$ such that $u_i(s') \neq 0, u_k(s') \neq 0$ we have

$$\frac{u_i(s)}{u_i(s')} = \frac{u_k(s)}{u_k(s')}.$$

A characterization of WCGIs

The game	The set of strategies	Conditions for $\lvert M\rvert, \lvert N\rvert$	Axioms
WCGC	M	$\lvert N\rvert = 1$ or $\lvert M\rvert = 1$	(P)
		$\lvert N\rvert = 2, \lvert M\rvert = 2$	(P), (PM)
		$\lvert N\rvert \geq 3, \lvert M\rvert = 2$	(P), (PM), (PS)
		$\lvert N\rvert \geq 2, \lvert M\rvert \geq 3$	(P), (PM), (PS), (IIC)
	2^M	$\lvert N\rvert = 1, \lvert M\rvert = 1$	(NN)
		$\lvert N\rvert = 1, \lvert M\rvert \geq 2$	(NN), (RA)
		$\lvert N\rvert \geq 2, \lvert M\rvert = 1$	(NN), (PS)
		$\lvert N\rvert \geq 2, \lvert M\rvert \geq 2$	(NN), (RA), (PS), (PM)
			(NN), (T), (PS), (PM), (IIC)