A complete t-intersection theorem for families of spanning trees

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t-intersecting theorems

Theorem 1 (Erdős, Ko, Rado)

Let n and k be positive integers with $n \geq 2k$. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting, that is, $F \cap F' \neq \emptyset$ for all $F, F' \in \mathcal{F}$. Then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

There are many variations:

- Instead of $\binom{[n]}{k}$ we can consider $2^{[n]}$, linear spaces over finite fields, permutations, trees etc
- ‘intersecting’ can be substituted with ‘t-intersecting’

Since permutations can be thought of as perfect matchings in complete bipartite graphs, our case is somewhat similar.

Complete t-intersecting theorems

Theorem 2 (Ahlswede, Khachatrian)

Let n, k, t be positive integers. Let $M(n,k,t)$ denote the maximum size of t-intersecting families in $\binom{[n]}{k}$. Then $M(n,k,t) = |\mathcal{F}^{(k)}(n,t,r)|$ for some $0 \le r \le \frac{n-t}{2}$, where

$$\mathcal{F}^{(k)}(n,t,r) = \left\{ F \in \binom{[n]}{k} : |F \cap [t+2r]| \ge t+r \right\}$$

Moreover, if $t \ge 2$, then $F^{(k)}(n,t,r)$ are the unique type of configurations up to isomorphism.

Note: an exact dependency on r is known.
We also aim for a structural result and in our proof families similar to $\mathcal{F}^{(k)}(n,t,r)$ arise.

Problem setting

Let $\mathcal{T}_n$ be the family of all spanning trees of K_n.
We say that a family $\mathcal{F} \subset \mathcal{T}_n$ is *t-intersecting* if for all $A, B \in \mathcal{F}$ the trees A and B share at least t edges.

Main question

What are the size and the structure of a maximal t-intersecting family $\mathcal{F} \subset \mathcal{T}_n$?

Previous results

Theorem 3 (Frankl, Hurlbert, Ihringer, Kupavskii, Lindzey, Meagher, Pantangi)

For $n \geq 2^{19}$, $t \geq 1$, let $\mathcal{F}$ be a t-intersecting family of spanning trees on n vertices.

1. *If $t = 1$, then*

$$|\mathcal{F}| \leq 2n^{n-3} + n - 2.$$

   Moreover, equality holds if and only if $\mathcal{F}$ consists of all stars plus all trees containing a fixed edge.

2. *If $1 < t \leq \frac{n}{4032 \log_2 n}$, then*

$$|\mathcal{F}| \leq 2^t n^{n-t-2}.$$

   Moreover, equality holds if and only if $\mathcal{F}$ consists of all trees containing a fixed set of t pairwise disjoint edges.

Main theorem

Theorem 4

There exists an absolute constant n_0 such that for all $n \geq n_0$ and $2 \leq t \leq n-2$ the following holds. If $\mathcal{F} \subset \mathcal{T}_n$ is a t-intersecting family then $|\mathcal{F}| \leq c_{n,t} n^{n-2-t}$, where $c_{n,t}$ is the largest possible product of $n-t$ positive integers with sum n. Moreover, the equality holds if and only if $\mathcal{F}$ is the family of all spanning trees containing a fixed forest F with t edges and connected components of almost equal sizes.

About the maximal example

Lemma 5 (Lu, Mohr, Székely)

Let F be a spanning forest in the K_n that has m connected components of sizes $q_1, q_2, \dots, q_m$. Then the number of spanning trees in K_n, that contain F is exactly

$$q_1 q_2 \cdots q_m \; n^{n-2-\sum_{i=1}^{m}(q_i-1)}.$$

Without loss of generality we may assume that $q_1 + q_2 + \ldots + q_m = n$. We search for extremal families among

$$\mathcal{U}_{n,t,r,F} = \{A \in \mathcal{T}_n \mid |A \cap F| \geq t + r\}.$$

which are natural counterparts of $\mathcal{F}^{(k)}(n,t,r)$ in the domain of trees.

Denote

$$\mathcal{A}_{n,t,r} = \max_{F-\text{forest of size } t+2r} |\mathcal{U}_{n,t,r,F}|$$
$$= \max_{F-\text{forest of size } t+2r} \{A \in \mathcal{T}_n \mid |A \cap F| \geq t+r\}.$$

We prove two lemmas which together show that only extremal trivial examples are possible.

Lemma 6

There is $n_0 \in \mathbb{N}$ and an absolute constant $\alpha < 1$ such that for any positive integers n, t, r such that $t \leq n-2$, $t+r \leq n-1$ and $n \geq n_0$ we have $\mathcal{A}_{n,t,r} \leq \alpha \mathcal{A}_{n,t,0}$ whenever we have either $r \geq 2$, or $r = 1$ and $t \geq n/2$.

Lemma 7

There is a positive constant n_0 such that for any positive integers n, t such that $t < n/2$ and $n > n_0$ we have $\mathcal{A}_{n,t,0} > \mathcal{A}_{n,t,1}$.

Comparison with permutations

Denote Σ_n to be the family of all permutations of some n-element set,

$$\mathcal{U}_k = \{\sigma \mathcal{A}_k \tau : \sigma, \tau \in \Sigma_n\}, \text{ where}$$

$$\mathcal{A}_k = \{\sigma \in \Sigma_n : \sigma(i) = i \text{ for at least } t + k \text{ indices } i \in [t + 2k]\}.$$

Theorem 8 (Kupavskii)

Let $\eta > 0$ be some constant and n, t be integers suzh that $n > (1 + \eta)t$ and $n \geq n_0(\eta)$. If $\mathcal{F} \subset \Sigma_n$ is t-intersecting then $|\mathcal{F}| \leq \max_{k \in \mathbb{N}} |\mathcal{A}_k|$. Moreover, if $\mathcal{F}$ is not fully contained in some $\mathcal{F} \in \mathcal{U}_k$, then $|\mathcal{F}| \leq (1 - 1/e + o(1)) \max_{k \in \mathbb{N}} |\mathcal{A}_k|$.

- In the permutation case more complicated maximal examples are possible;
- There is a stability result, i.e. all maximal families that are different from described above are smaller by a constant factor. It is not true for trees when t is slightly less than $n/2$.

Proof structure

We may assume $t > 2n^{1-\varepsilon/7}$ for some very small $\varepsilon > 0$. Let $\mathcal{F}$ be the maximal t-intersecting family.

1. For some $t' < t$ we find a t'-intersecting set $\mathcal{S}$ of forests of sizes smaller than $(1+\varepsilon)t$ such that $\mathcal{F}$, i.e. $\mathcal{F} \setminus \mathcal{F}[\mathcal{S}]$ is small;
2. Via simplification-peeling technique we analyse the structure of $\mathcal{S}$ and find a restriction X such that $\mathcal{F}(X)$ is 'dense' in $\mathcal{T}_n(X)$. Inside this $\mathcal{F}(X)$ we identify a very spread subfamily, and then via a greedy procedure we build a t-intersecting spread approximation $\mathcal{S}'$;
3. We apply a variant of the peeling-simplification process to $\mathcal{S}'$ in order to determine the most important 'layer'. As a result, we prove that all but $o(|\mathcal{F}|)$ sets of $\mathcal{F}$ lie in some $\mathcal{U}_{n,t,r,F}$. The last step is to show that in the extremal family the remainder must be empty;
4. We prove that in fact this remainder of size $o(|\mathcal{F}|)$ should be empty in extremal families.

Spread families

Definition 9

Given $r > 1$ a family $\mathcal{F} \subset 2^{[n]}$ is said to be *r-spread* if

$$|\mathcal{F}(X)| < r^{-|X|}|\mathcal{F}|$$

for all $X \subset [n]$.

For example, $\mathcal{T}_n$ is $n/2$-spread.
Moreover, let F be a spanning forest with t edges maximising $|\mathcal{F}(T)|$ among all spanning forests with t edges. Then $\mathcal{T}_n(F)$ is $n/2$-spread.
Proof: Let U be a spanning forest that is edge-disjoint from F.

$$\begin{aligned}|\mathcal{T}_n(F \sqcup U)| \leq & c_{n,t+|U|} n^{n-2-t-|U|} \leq c_{n,t} \cdot 2^{|U|} n^{n-2-t-|U|} \\ = & (n/2)^{-|U|} c_{n,t} n^{n-2-t} = (n/2)^{-|U|} |\mathcal{T}_n(F)|.\end{aligned}$$

The second inequality will be verified later.

$\mathcal{T}_n$ has spread properties

Lemma 10

Let n, t be nonnegative integers such that $t \le n-2$. Then

$$1/n \le c_{n,t+1}/c_{n,t} \le 2.$$

Proof: Recall that $c_{n,t}$ can be thought of as of the product of $n-t$ integer factors of almost equal values with sum n. Consider a sequence of such factors $q_1 \le q_2 \le \ldots \le q_{n-t}$. For $i = 1, \ldots, n-t-1$ put $p_i = q_{i+1}$. Then for $j = 1, \ldots, q_1$ increase p_j by one where the numeration of p_i's is considered to be cyclic, $p_{n-t} = p_1$. It means that if $q_1 > n-t-1$, then there will be some p_i's that we increase by 1 more then once. Then

$$\frac{c_{n,t+1}}{c_{n,t}} = \frac{p_1 \cdot \ldots \cdot p_{n-t-1}}{q_1 \cdot \ldots \cdot q_{n-t}} \le \frac{\left(1+\frac{1}{k}\right)^k}{k} \le 2;$$

$$\frac{c_{n,t+1}}{c_{n,t}} = \frac{p_1 \cdot \ldots \cdot p_{n-t-1}}{q_1 \cdot \ldots \cdot q_{n-t}} \ge \frac{1}{k} \ge \frac{1}{n}.$$

Spread subfamilies of big families

Observation 11

If for some $\alpha > 1$ we have a family $\mathcal{F} \subset \binom{[n]}{\leq k}$ such that $|\mathcal{F}| > \alpha^k$, then there is a set X such that $|\mathcal{F}(X)| > 1$ and $\mathcal{F}(X)$ is α-spread.

Proof: If $\mathcal{F}$ is α-spread then we are done. Otherwise, find an inclusion-maximal set X such that $|\mathcal{F}(X)| \geq \alpha^{-|X|}|\mathcal{F}|$. Note that in this case $|\mathcal{F}(X)| \geq \alpha^{-k}|\mathcal{F}| > 1$. Assume $\mathcal{F}(X)$ is not spread, then there is a nonempty set Y disjoint from X such that $|\mathcal{F}(X \cup Y)| \geq \alpha^{-|Y|}|\mathcal{F}(X)| \geq \alpha^{-|X|-|Y|}|\mathcal{F}|$. But then we should have chosen $Y \cup X$ in place of X which was supposed to be inclusion-maximal.

Observation 12

Fix $\alpha > 1$ and consider a family $\mathcal{F}$. If X is an inclusion-maximal set with the property that $\mathcal{F}(X) \geq \alpha^{-|X|}|\mathcal{F}|$, then $\mathcal{F}(X)$ is α-spread.

The proof is similar to above.

The Spread Lemma

Theorem 13 (Alweiss, sharpening due to Hu)

If for some $n, k, r \geq 1$ a family $\mathcal{F} \subset \binom{[n]}{\leq k}$ is r-spread and W is an $(\beta\delta)$-random subset of $[n]$, then

$$\Pr[\exists F \in \mathcal{F} \ : \ F \subset W] \geq 1 - \left(\frac{2}{\log_2(r\delta)}\right)^{\beta} k.$$

We will also use an application of this theorem to find disjoint sets in spread families.

Lemma 14

Let $\mathcal{G}_1, \mathcal{G}_2 \subset 2^{[m]}$ be R-spread uniform families of uniformity at most k. Suppose that $R > 2^5 \log_2 2k$. Then there exist disjoint sets $G_1 \in \mathcal{G}_1, G_2 \in \mathcal{G}_2$.

Disjoint sets in spread families

Proof: Let us put $\beta = \log_2(2k)$ and $\delta = (2\log_2(2k))^{-1}$. Note that $\beta\delta = \frac{1}{2}$ and $R\delta > 2^4$ by our choice of R. Theorem 13 implies that a $\frac{1}{2}$-random subset W_i of $[m]$ contains a set from $\mathcal{G}_i$ with probability strictly bigger than

$$1 - \Big(\frac{2}{\log_2 2^4}\Big)^{\log_2(2k)} k = 1 - 2^{-\log_2(2k)} k = \frac{1}{2}.$$

Note that $[n] \setminus W$ is also a $\frac{1}{2}$-random subset of $[m]$. A simple union bound implies that with positive probability there are $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$ such that $G_1 \subset W$, $G_2 \subset [m] \setminus W$. Thus there are disjoint $G_1 \in \mathcal{G}_1, G_2 \in \mathcal{G}_2$.

First spread approximation

We find a family of spanning forests $\mathcal{S}$ such that $\mathcal{R} = \mathcal{F} \setminus \mathcal{F}[\mathcal{S}]$ is small (i.e. $|\mathcal{R}| < n^{-\frac{1}{3}\varepsilon t} \cdot c_{n,t} n^{n-t-2}$) and decompose

$$\mathcal{F} = \mathcal{R} \sqcup \bigsqcup_{S \in \mathcal{S}} \mathcal{F}_S[S]$$

in such a way that $\mathcal{F}_S[S]$ is $n^{\varepsilon/2}$-spread.

Proof idea: Greedy algorithm, at each step find a maximal S_i that $|\mathcal{F}^i(S_i)| \geq (n^{\varepsilon/2})^{-|S_i|}|\mathcal{F}^i|$, then delete $\mathcal{F}^i[S_i]$ from $\mathcal{F}^i$ to obtain $\mathcal{F}^{i+1}$. Observation 12 together with spreadness of $\mathcal{T}_n$ guarantee that the remainder is small. Using the same argument as in Lemma 14 we prove that $\mathcal{S}$ is t'-intersecting for t' slightly less than t.

Peeling procedure

Observation 15

For any positive integers n, q and a t-intersecting family $\mathcal{S} \subset \mathcal{T}_n^{\leq q}$, there exists a maximal t-intersecting family $\mathcal{H}$ such that for every $H \in \mathcal{H}$ there is $S \in \mathcal{S}$ such that $H \subseteq S$. Consequently, for any family $\mathcal{F} \subset \mathcal{T}_n$ we have $\mathcal{F}[\mathcal{S}] \subset \mathcal{F}[\mathcal{H}]$.

Consider a t-intersecting family $\mathcal{S} \subset \mathcal{T}_n^{\leq q}$. Let us iteratively define the following series of families.

1. Put $\mathcal{H}_{q-t} = \mathcal{S}$.
2. For $k = q-t, q-t-1, \dots, 0$ we put $\mathcal{W}_k = \mathcal{H}_k \cap \binom{[n]}{t+k}$ and let $\mathcal{H}_{k-1}$ be the family given by Observation 15 when applied to $\mathcal{H}_k \setminus \mathcal{W}_k$ playing the role of $\mathcal{S}$.

Thus, we 'peel' the sets of the largest uniformity in $\mathcal{H}_k$ and replace the resulting family by a maximal t-intersecting family using Observation 15

Dealing with the remainder if it is small

By now we know that all but $o(|\mathcal{F}|)$ sets are in $\mathcal{F}[F]$ for some spanning forest F with t edges. We aim to prove that in fact the remainder $\mathcal{R} = \mathcal{F} \setminus \mathcal{F}[F]$ is empty. Note that that even one element $T \in \mathcal{R}$ makes us delete something from $\mathcal{F}[F]$ (i.e. trees that contain F but do not t-intersect T)
If $|\mathcal{R}| < n^{-\frac{1}{3}\varepsilon t} \cdot c_{n,t} n^{n-t-2} < n^{n-t-27}$, we can use the following lemma to complete the proof.

Lemma 16

Let F be a forest on n vertices with t edges, $t \leq n - 55$, T be a spanning tree on n vertices which is not a star, $F \not\subset T$. Then the number of spanning trees in $\mathcal{T}_n$ that contain F and avoid edges of T outside of F is at least n^{n-t-27}.

Dealing with the remainder if it is big

If $|\mathcal{R}| > n^{-\frac{1}{3}\varepsilon t} \cdot c_{n,t} n^{n-t-2}$, we construct a spread approximation of $\mathcal{R}$, i.e.

$$\mathcal{R} = \mathcal{R}' \sqcup \bigsqcup_{S \in \mathcal{S}} \mathcal{R}_S[S].$$

We fix some $S \in \mathcal{S}$. A quantitive argument guarantees that we can remove from $\mathcal{F}$ all sets that have at least t common elements with S, and $\mathcal{F}$ will still be big. Then we find a spread subfamily in $\mathcal{F}_1 \subset \mathcal{F}(F)$ and apply Lemma 14 to $\mathcal{F}_1$ and $\mathcal{R}_S(S)$, which completes the proof.

One more problem

In our proof we needed a lower bound on the number of spanning trees that contain a fixed forest F and avoid a tree T outside of F. In a more general setting:

Question

Let G be an arbitrary n-vertex graph, F be its spanning forest. Denote $\mathcal{T}_n[G, F]$ the family of all spanning trees in G that contain F.
Is there a nice formula for $\mathcal{T}_n[G, F]$?
What is

$$\min_{G:|E(G)|=\binom{n}{2}-u, F:|E(F)|=t} |\mathcal{T}_n[G, F]|?$$

We know that in case of $t = 1$ the probability that a uniformly random spanning tree contains a fixed edge e is its leverage score

$$l_e = w_e R_{eff}(e).$$

Литература

R. Alweiss, S. Lovett, K. Wu, and J. Zhang, *Improved bounds for the sunflower lemma*, arXiv:1908.08483, (2019)

A. Kupavskii, *An almost complete t-intersection theorem for permutations, arXiv preprint arXiv:2405.07843*, (2024)

P. Frankl, G. Hurlbert, F. Ihringer, A. Kupavskii, N. Lindzey, K. Meagher, V. R. T. Pantangi, *Intersecting Families of Spanning Trees, arXiv preprint arXiv:2502.08128*, (2024).