A second-order numerical method for fourth-order neutral Volterra integro-differential equation

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The V All-Russian Scientific Conference
Autumun Mathematical Readings in Adygea
Adyghe State University, Maykop, Adygea, Russian Federation

October 9-14, 2025

Overview

Düzce

Düzce is a city located in the northwestern part of Turkey, between Istanbul and Ankara.

It became the 81st province of Turkey in 1999 after the major earthquake that struck the region.

The city is surrounded by natural beauty, with forests, mountains, rivers, and the Black Sea coastline.

Düzce is well-known for its rich greenery, waterfalls, and outdoor sports opportunities such as rafting and hiking.

Düzce

Aydınpınar Waterfall

Düzce

Akçakoca

Düzce

Ceneviz Castle

Düzce

Efteni Lake

Düzce

Güzeldere Waterfall

Düzce

Samandere Waterfall

Sarıkaya Cave

The Aim Of This Study

- This work's objective is to introduce a numerical method for solving a neutral Volterra integro-differential equation (NVIDE) that involves fourth and second-order derivatives.
- First, the stability properties of exact solution are analyzed.
- To solve this problem numerically, on the uniform mesh, the finite difference method including the composite trapezoidal rule for the integral part of the equation is used.
- The method has been demonstrated to be second-order convergence in the discrete maximum norm.
- In order to verify the efficiency of the suggested method, numerical examples are given.

A Brief Overview of Integro-Differential Equations

There are numerous uses for integral-differential equations (IDEs) in both science and engineering.

For example, the behavior of a kinetic model of the interactions between the tumor, host environment, and immune system is qualitatively analyzed in [1]. The general equation gives a thorough explanation of how the number densities change over time.

Arlotti, L., Lachowicz, M.: Qualitative analysis of a nonlinear integro-differential equation modeling tumor-host dynamics. *Math. Comput. Modelling* **23**(6), 11–19 (1996)

Bellomo and Forni [2] proposed that the system consists of five different interacting populations, P_i, $i = 1, \dots, 5$. Particularly, P_1 represents tumor cells, P_2 represents host environment cells, P_3 represents polymorpho nuclear leukocytes, P_4 represents lymphocytes, and P_5 represents macrophages.

The variable u, which falls within the interval $[0, 1]$ and is called *activation*, defines the condition of cells that belong to the five populations.

Bellomo, N., Forni, G.: Dynamics of tumor interaction with host immune system. *Math. Comput. Modelling* **20**(1), 107–122 (1994)

Volterra integro-differential equations (VIDEs) are typically the outcome of applying mathematical modeling to real-world issues.

VIDEs have numerous uses in both science and engineering, for example, biology, mechanics, chemistry, biomedicine (see e.g., [3, 4, 5, 6]).

Assari, P.: The thin plate spline collocation method for solving integro-differential equations arisen from the charged particle motion in oscillating magnetic fields. *Engineering Computations* **35**(4), 1706–1726 (2018)

Ghosh, B., Mohapatra, J.: An iterative difference scheme for solving arbitrary order nonlinear Volterra integro-differential population growth model. *J. Anal.* **32**(1), 57–72 (2024)

De Gaetano, A., Arino, O.: Mathematical modelling of the intravenous glucose tolerance test. *J. Math. Biol.* **40**, 136–168 (2000)

Khaldi, K.E., Rabiei, N., Saleeby, E.G.: On modeling column crystallizers and a Hermite predictor-corrector scheme for a system of integro-differential equations. *Int. J. Nonlinear Sci. Numer. Simul.* **24**(2), 489–530 (2023)

Yang et al. [7] studied numerical method that called multistep pseudo-spectral continuous Galerkin approach to solve first-order VIDE. Their method has spectral accuracy.

Yang, Y., Yao, P., Tohidi, E.: Convergence analysis of an efficient multistep pseudo-spectral continuous Galerkin approach for solving Volterra integro-differential equations. *Appl. Math. Comput.* **494**, 129284 (2025)

In [8], the authors for the singularly perturbed VIDE utilized the finite difference method, which depends on a uniform mesh, achieved through the use of the composite right-side rectangle rule for the integral part and the implicit difference rules for the differential part. This method yields first-order uniform convergence

Temel, Z., Cakir, M.: A numerical method for solving linear first-order Volterra integro-differential equations with integral boundary condition. *Appl. Appl. Math.* **19**(2) (2024)

Liu et al. [9] investigated numerical approach for VIDE with weakly singular kernel. The method is based on finite difference scheme on a graded mesh, and it is shown that the method has second-order accuracy.

Liu, L.-B., Ye, L., Bao, X., Zhang, Y.: A second order numerical method for a Volterra integro-differential equation with a weakly singular kernel. *Netw. Heterog. Media* **19**(2), 740–752 (2024)

Also, in [10], investigated a method to solve numerically the fourth-order VIDE that was based on the decomposition method and Green's function. It has been demonstrated that an accurate approximation of the solution can be obtained with only a two-term series solution.

Singh, R., Wazwaz, A.-M.: Numerical solutions of fourth-order Volterra integro-differential equations by the Green's function and decomposition method. *Math. Sci.* **10**, 159–166 (2016)

Afrouzi et al. [11] used modified homotopy perturbation method to obtain numerical solution for fourth-order VIDE.

Afrouzi, G., Ganji, D., Hosseinzadeh, H., Talarposhti, R.: Fourth order Volterra integro-differential equations using modified homotopy perturbation method. *J. Math. Comput. Sci.* **3**(2), 179–191 (2011)

[12] proposed Integrated Simpson's Collocation Method for fourth-order linear and nonlinear VIDE. The suggested approach and the exact solution had a very good comparison.

Jimoh, A.K.: Integrated Simpson's collocation method for solving fourth order Volterra integro-differential equations. *Glob. J. Pure Appl. Math.* **16**(5), 657–673 (2020)

Consequently, many studies can be given as examples of numerical solutions of VIDEs, such as [13, 14, 15].

Amirali, I., Durmaz, M.E., Amiraliyev, G.M.: A numerical approach for solving Volterra delay integro-differential equation. *Miskolc Math. Notes* **25**(2), 553–562 (2024)

Aourir, E., Dastjerdi, H.L.: Numerical computational technique for solving Volterra integro-differential equations of the third kind using meshless collocation method. *J. Comput. Appl. Math.* **457**, 116294 (2025)

Brunner, H.: Collocation methods for nonlinear Volterra integro-differential equations with infinite delay. *Math. Comput.* **53**(188), 571–587 (1989)

In this work we investigate fourth-order NVIDE. The neutral type is a result of the highest derivative under the integral sign. Numerical solutions to VIDEs of the neutral type have attracted the attention of many authors.

Shi et al. [16] studied the Chebyshev spectral collocation method for high-order nonlinear VIDEs of neutral type.

This numerical method is shown to converge exponentially.

Shi, X., Huang, F., Hu, H.: Convergence analysis of spectral methods for high-order nonlinear Volterra integro-differential equations. *Comput. Appl. Math.* **38**(2), 56 (2019)

In [17], a difference scheme approach on a uniform mesh is used to solve NVIDE.
It is shown that the finite difference method has second-order accuracy.

Amirali, I., Fedakar, B., Amiraliyev, G.M.: Second-order numerical method for a neutral Volterra integro-differential equation. *J. Comput. Appl. Math.* **453**, 116160 (2025)

Sedaghat et al. [18], in order to approximate Volterra functional integro-differential equations of neutral type, examine the convergence properties of the spectral approach, and exponential order of convergence is obtained.

Sedaghat, S., Ordokhani, Y., Dehghan, M.: On spectral method for Volterra functional integro-differential equations of neutral type. *Numer. Funct. Anal. Optim.* **35**(2), 223–239 (2014)

Wang [19] examined the numerical stability of one-leg approaches for nonlinear NVIDE with constant delay.

Wang, W.: Nonlinear stability of one-leg methods for neutral Volterra delay integro-differential equations. *Math. Comput. Simul.* **97**, 147–161 (2014)

Other studies can be found in [20, 21, 22].

Wei, Y., Chen, Y.: Legendre spectral collocation method for neutral and high-order Volterra integro-differential equation. *Appl. Numer. Math.* **81**, 15–29 (2014)

Fedakar, B., Amirali, I., Durmaz, M.E., Amiraliyev, G.M.: Numerical solutions for second-order neutral Volterra integro-differential equations: Stability analysis and finite difference method. *J. Comput. Appl. Math.* **459**, 116371 (2025)

Reutskiy, S.Y.: The backward substitution method for multipoint problems with linear Volterra-Fredholm integro-differential equations of the neutral type. *J. Comput. Appl. Math.* **296**, 724–738 (2016)

To the best of our knowledge, the use of finite difference schemes for approximating solutions of fourth-order NVIDEs has not been explored. In this work, we introduce a numerical method that achieves second-order convergence.
These aspects contribute to the effectiveness of the study.

We consider the initial-value problem for neutral Volterra integro-differential equation (NVIDE) in the domain $\Omega = (0, T]$:

$$u^{(4)}(t) + a(t)\, u''(t) + b(t)\, u(t) - \int_0^t \left[K_1(t,s)\, u^{(4)}(s) + K_2(t,s)\, u''(s) + K_3(t,s)\, u(s)\right] ds = f(t), \tag{1}$$

$$u^{(m)}(0) = \mu_m, m = 0, 1, 2, 3, \tag{2}$$

where $a(t)$, $b(t)$, $f(t)$ $\left(t \in \bar{\Omega} \equiv [0, T]\right)$ and $K_i(t,s)\,, i = 1, 2, 3, \left((t,s) \in \bar{\Omega}^2\right)$are given sufficiently smooth functions that satisfy certain regularity conditions to be specified, μ_m $(m = 0, 1, 2, 3)$ are given constants.

Differential Problem

This section deals with the stability properties of the exact solution of the problem (1)-(2) in relation to the right-hand side and initial conditions. Here we will use

$$\|g\|_{\infty} \equiv \|g\|_{\infty,\bar{\Omega}} := \max_{\bar{\Omega}} |g(t)| .$$

Theorem 1

For $a(t)$,$b(t)$,$f(t) \in C[0, T]$, $K_i(t,s)$, $i = 1,2,3 \in C[0, T]^2$, the solution of problem (1)-(2) satisfies the following stability bound

$$|u(t)| \leq \left(|\mu_0| + T|\mu_1| + T^2 d_1\right) e^{T^2 d_2 t}, \quad 0 < t \leq T, \tag{3}$$

where

$$\begin{aligned}
d_1 &= \left(1 + c_0 T e^{c_0 T}\right)\left[|\mu_2| + T|\mu_3| + T^2\left(1 + T\bar{K}_1 e^{\bar{K}_1 T}\right)\|f\|_\infty\right], \\
d_2 &= \left(1 + c_0 T e^{c_0 T}\right)\left[T\|b\|_\infty + T^2\left(\bar{K}_3 + \bar{K}_1 e^{\bar{K}_1 T}\|b\|_\infty + T\bar{K}_1\bar{K}_3 e^{\bar{K}_1 T}\right)\right], \\
c_0 &= T\|a\|_\infty + T^2\left(\bar{K}_2 + \bar{K}_1 e^{\bar{K}_1 T}\|a\|_\infty + T\bar{K}_1\bar{K}_2 e^{\bar{K}_1 T}\right), \\
\bar{K}_i &= \max_{\bar{\Omega}^2}|K_i(t,s)|, \quad i = 1,2,3.
\end{aligned}$$

Proof of Theorem 1

Showing $\delta(t) = \left|u^{(4)}(t)\right|$, from (1) we get

$$\delta(t) \le \rho(t) + \bar{K}_1 \int\limits_0^t \delta(s)ds, \tag{4}$$

where

$$\rho(t) = \|a\|_\infty \left|u''(t)\right| + \|b\|_\infty |u(t)| + \int\limits_0^t \left(\bar{K}_2 \left|u''(s)\right| + \bar{K}_3 |u(s)|\right) ds + \|f\|_\infty .$$

Using Gronwall's inequality from (4), we obtain

$$\delta(t) \le \rho(t) + \bar{K}_1 e^{\bar{K}_1 T} \int\limits_0^t \rho(s)ds.$$

Hence

$$
\begin{aligned}
\left|u^{(4)}(t)\right| \leq & \|a\|_{\infty}\left|u''(t)\right| + \|b\|_{\infty}|u(t)| + \int_0^t \left(\bar{K}_2\left|u''(s)\right| + \bar{K}_3|u(s)|\right)ds \\
& + \|f\|_{\infty} + \bar{K}_1 e^{\bar{K}_1 T}\int_0^t \left[\begin{array}{l} \|a\|_{\infty}|u''(s)| + \|b\|_{\infty}|u(s)| \\ + \int_0^s \left(\bar{K}_2|u''(\xi)| + \bar{K}_3|u(\xi)|\right)d\xi + \|f\|_{\infty}\end{array}\right]ds \\
& \leq \|a\|_{\infty}\left|u''(t)\right| + \|b\|_{\infty}|u(t)| + \left(1 + T\bar{K}_1 e^{\bar{K}_1 T}\right)\|f\|_{\infty} \\
& + \int_0^t \left[\begin{array}{l} \left(\bar{K}_2 + \bar{K}_1 e^{\bar{K}_1 T}\|a\|_{\infty} + T\bar{K}_1\bar{K}_2 e^{\bar{K}_1 T}\right)|u''(s)| \\ + \left(\bar{K}_3 + \bar{K}_1 e^{\bar{K}_1 T}\|b\|_{\infty} + T\bar{K}_1\bar{K}_3 e^{\bar{K}_1 T}\right)|u(s)|\end{array}\right]ds.
\end{aligned} \tag{5}
$$

Note that for any $g \in C^4\left[0, T\right]$,

$$g\left(t\right) = g\left(0\right) + tg'\left(0\right) + \int\limits_0^t \left(t-s\right) g''\left(s\right) ds,$$

thereby

$$\left|g\left(t\right)\right| \le \left|g\left(0\right)\right| + T\left|g'\left(0\right)\right| + T\int\limits_0^t \left|g''\left(s\right)\right| ds. \tag{6}$$

Using this inequality for $g\left(t\right) = u''\left(t\right)$, from (5) it follows that

$$\begin{aligned}
\left|u''\left(t\right)\right| \le{} & \left|\mu_2\right| + T\left|\mu_3\right| \\
& + T\int\limits_0^t \left[\begin{array}{l} \|a\|_\infty \left|u''\left(s\right)\right| + \|b\|_\infty \left|u\left(s\right)\right| + \left(1 + T\bar{K}_1 e^{\bar{K}_1 T}\right)\|f\|_\infty \\ + \int\limits_0^s \left[\begin{array}{l} \left(\bar{K}_2 + \bar{K}_1 e^{\bar{K}_1 T}\|a\|_\infty + T\bar{K}_2\bar{K}_1 e^{\bar{K}_1 T}\right)\left|u''\left(\xi\right)\right| \\ + \left(\bar{K}_3 + \bar{K}_1 e^{\bar{K}_1 T}\|b\|_\infty + T\bar{K}_1\bar{K}_3 e^{\bar{K}_1 T}\right)\left|u\left(\xi\right)\right| \end{array}\right] d\xi \end{array}\right] ds \\
\le{} & \sigma\left(t\right) + c_0 \int\limits_0^t \left|u''\left(s\right)\right| ds
\end{aligned}$$

where

$$\sigma(t) = |\mu_2| + T|\mu_3| + T^2\left(1 + T\bar{K}_1 e^{\bar{K}_1 T}\right)\|f\|_\infty$$
$$+ \left[T\|b\|_\infty + T^2\left(\bar{K}_3 + \bar{K}_1 e^{\bar{K}_1 T}\|b\|_\infty + T\bar{K}_1\bar{K}_3 e^{\bar{K}_1 T}\right)\right]\int_0^t |u(s)|ds.$$

After using the Gronwall's inequality, we have that

$$\left|u''(t)\right| \leq \sigma(t) + c_0 e^{c_0 T}\int_0^t \sigma(s)ds,$$

Therefore

$$\left|u''(t)\right| \leq \left(1 + c_0 T e^{c_0 T}\right)\sigma(t)$$
$$\leq d_1 + d_2\int_0^t |u(s)|ds. \tag{7}$$

Applying the inequality (6) for $g(t) = |u(t)|$ and using (7), we have

$$|u(t)| \leq |\mu_0| + T|\mu_1| + T\int_0^t |u''(s)|\, ds$$

$$\leq |\mu_0| + T|\mu_1| + T^2 d_1 + T^2 d_2 \int_0^t |u(s)|\, ds,$$

which after applying the Gronwall's inequality for $|u(t)|$ leads to (3).

Construction of difference scheme

Let ω_N be any uniform mesh on $\bar{\Omega}$:

$$\omega_N = \left\{ t_i = ih, i = 1, 2, \ldots, N-1, h = \frac{T}{N} \right\}, \quad \bar{\omega}_N = \omega_N \cup \{t_0 = 0, t_N = T\}.$$

The following notations are used for any mesh function $v(t)$ defined on $\bar{\omega}_N$.

$$v_i = v(t_i), \; v_{t,i} = \frac{v_{i+1} - v_i}{h}, \; v_{\bar{t},i} = \frac{v_i - v_{i-1}}{h}, v_{\bar{t}t,i} = \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2},$$
$$v_{\bar{t}tt,i} = \frac{v_{i+2} - 3v_{i+1} + 3v_i - v_{i-1}}{h^3}, v_{\bar{t}t\bar{t}t,i} = \frac{1}{h^4}\left(v_{i+2} - 4v_{i+1} + 6v_i - 4v_{i-1} + v_{i-2}\right)$$

and the discrete norms

$$\|v\|_\infty \equiv \|v\|_{\infty,\bar{\omega}_N} := \max_{0 \le i \le N} |v_i|, \quad \|v\|_1 \equiv \|v\|_{1,\omega_N} := h \sum_{i=2}^{N-2} |v_i|.$$

Let $t = t_i$ in (1)

$$u^{(4)}(t_i) + a(t_i)\,u''(t_i) + b(t_i)\,u(t_i) + \int_0^{t_i} \left[K_1(t_i, s)\,u^{(4)}(s) + K_2(t_i, s)\,u''(s) + K_3(t_i, s)\,u(s)\right] ds = f(t_i),\ 2 \le i \le N-2.$$

Replacing each of $u^{(4)}(t_i)$ and $u''(t_i)$ with appropriate difference derivative we get

$$\begin{aligned} u^{(4)}(t_i) + a(t_i)\,u''(t_i) + b(t_i)\,u(t_i) &= u_{\bar{t}\bar{t}tt,i} + a(t_i)\,u_{\bar{t}t,i} + b(t_i)\,u(t_i) \\ &\quad + R_i^{(1)} + R_i^{(2)},\ 2 \le i \le N-2, \end{aligned} \tag{8}$$

where

$$\begin{aligned} R_i^{(1)} &= -\frac{h^2}{6} u^{(6)}\left(\xi_i^{(1)}\right), \quad \xi_i^{(1)} \in (t_{i-2}, t_{i+2}), \\ R_i^{(2)} &= -\frac{h^2}{12} u^{(4)}\left(\xi_i^{(2)}\right), \quad \xi_i^{(2)} \in (t_{i-1}, t_{i+1}). \end{aligned}$$

Applying the composite trapezoidal rule on $[0, t_i]$ for integral component containing $K_1(t_i, s)\,u^{(4)}(s)$, we get the following relation:

$$\int_0^{t_i} K_1(t_i, s) u^{(4)}(s)\, ds = \sum_{j=0}^{i} \hbar_j K_1(t_i, t_j)\, u^{(4)}(t_j) + \bar{R}_i^{(3)},$$

$$\hbar_0 = \hbar_i = \frac{h}{2}, \quad \hbar_j = h, \quad j = 1, 2, \ldots, i-1$$

where

$$\bar{R}_i^{(3)} = -\frac{h^2}{12} t_i \frac{d^2}{ds^2}\left(K_1\left(t_i, \xi_i^{(3)}\right) u^{(4)}\left(\xi_i^{(3)}\right)\right), \quad \xi_i^{(3)} \in (t_0, t_i)\,.$$

For the approximating $u^{(4)}(t_j)$ we will utilize

$$u^{(4)}(t_0) = \frac{u(t_4) - 4u(t_3) + 6u(t_2) - 4u(t_1) + u(t_0)}{h^4} + R_0^{(1)} = u_{\bar{t}t\bar{t}t,2} + R_0^{(1)},$$

where

$$R_0^{(1)} = -2hu^{(5)}\left(\xi_0^{(1)}\right), \xi_0{}^{(1)} \in (t_0, t_4),$$

$$u^{(4)}(t_1) = \frac{u(t_0) - 4u(t_1) + 6u(t_2) - 4u(t_3) + u(t_4)}{h^4} + R_1^{(1)} = u_{\bar{t}t\bar{t}t,2} + R_1^{(1)}, \tag{9}$$

$$R_1^{(1)} = -hu^{(5)}\left(\xi_1^{(1)}\right), \quad \xi_1^{(1)} \in (t_0, t_4)$$

and

$$u^{(4)}(t_j) = u_{\bar{t}t\bar{t}t,j} + R_j^{(1)} \text{ for } j \geq 2.$$

Thus, for the numerical approximation of the integral term comprising $K_1(t_i, s)$, it is obtained that

$$\int_0^{t_i} K_1(t_i, s) u^{(4)}(s)\, ds = \sum_{j=0}^{i} \hbar_j K_1(t_i, t_j) D^4 u_j + R_i^{(3)}, \tag{10}$$

here

$$D^4 u_j = \begin{cases} u_{\bar{t}\bar{t}\bar{t}t,2}, & \text{for } j = 0, 1, \\ u_{\bar{t}\bar{t}\bar{t}t,j}, & \text{for } 2 \le j \le i \end{cases}$$

and

$$R_i^{(3)} = \bar{R}_i^{(3)} + \sum_{j=0}^{i} \hbar_j K_1(t_i, t_j) R_j^{(1)}.$$

By repeating this process for the integral part, which encompasses $K_2\left(t_i\ , s\right)$, the composite trapezoidal rule gives:

$$\int_0^{t_i} K_2\left(t_i, s\right) u''\left(s\right) ds = \sum_{j=0}^{i} \hbar_j K_2\left(t_i, t_j\right) u''\left(t_j\right) + \bar{R}_i^{(4)}$$

with

$$\hbar_0 = \hbar_i = \frac{h}{2}, \quad \hbar_j = h, \quad j = 1, 2, \ldots, i-1$$

and

$$\bar{R}_i^{(4)} = -\frac{h^2 t_i}{12} \frac{d^2}{ds^2}\left(K_2\left(t_i, \xi_i^{(4)}\right) u''\left(\xi_i^{(4)}\right)\right), \quad \xi_i^{(4)} \in \left(t_0, t_i\right). \tag{11}$$

Using the correlation

$$u''(t_0) = \frac{u_2 - 2u_1 + u_0}{h^2} + R_0^{(2)} = u_{\bar{t}t,1} + R_0^{(2)} \tag{12}$$

where

$$R_0^{(2)} = -hu'''\left(\xi_0^{(2)}\right), \quad \xi_0^{(2)} \in (t_0, t_2), \tag{13}$$

we get

$$\int_0^{t_i} K_2(t_i, s)u''(s)\,ds = \sum_{j=0}^{i} \hbar_j K_2(t_i, t_j)\, D^2 u_j + R_i^{(4)}, \tag{14}$$

in which

$$D^2 u_j = \begin{cases} u_{\bar{t}t,j}, & \text{for } 1 \le j \le i, \\ u_{\bar{t}t,1}, & \text{for } j = 0, \end{cases}$$

$$R_i^{(4)} = \bar{R}_i^{(4)} + \sum_{j=0}^{i} \hbar_j K_2(t_i, t_j)\, R_j^{(2)}. \tag{15}$$

Next, applying the composite trapezoidal rule containing $K_3\left(t_i, s\right)$ we have

$$\int_0^{t_i} K_3\left(t_i, s\right) u(s)\, ds=\sum_{j=0}^{i} \hbar_j K_3\left(t_i, t_j\right) u_j+R_i^{(5)}, \tag{16}$$

$$R_i^{(5)}=-\frac{h^2}{12} t_i \frac{d^2}{d s^2}\left(K_3\left(t_i, \xi_i^{(5)}\right) u\left(\xi_i^{(5)}\right)\right), \quad \xi_i^{(5)} \in\left(t_0, t_i\right).$$

The following exact relation for (1) is yielded from the relations (10), (14), and (16):

$$u_{\bar{t}t\bar{t}t,i} + a_i u_{\bar{t}t,i} + b_i u_i - \sum_{j=0}^{i} \hbar_j \left[K_1 \left(t_i, t_j \right) D^4 u_j + K_2 \left(t_i, t_j \right) D^2 u_j + K_3 \left(t_i, t_j \right) u_j \right] + R_i$$

$$= f_i, \;\; 2 \leq i \leq N-2 \qquad (17)$$

with truncation error

$$R_i = R_i^{(1)} + R_i^{(2)} + R_i^{(3)} + R_i^{(4)} + R_i^{(5)}. \qquad (18)$$

Our aim now is to approximate the initial conditions. Because of

$$u_{t,0} = u'(0) + \frac{h}{2}u''(0) + \frac{h^2}{6}u'''(\xi_2), \quad 0 < \xi_2 < h,$$

then using also (2), we get

$$u_{t,0} = \bar{\mu}_1 + r^{(0)} \tag{19}$$

here

$$\bar{\mu}_1 = \mu_1 + \frac{h}{2}\mu_2, \quad r^{(0)} = \frac{h^2}{6}u'''(\xi_2), \quad 0 < \xi_2 < h. \tag{20}$$

Furthermore, utilizing the identity

$$u_{\bar{t}t,1} = u''(0) + hu'''(0) + r^{(1)},$$

$$r^{(1)} = \frac{7}{12}h^2 u^{(4)}(\xi_3) \tag{21}$$

and (2) we have

$$u_{\bar{t}t,1} = \bar{\mu}_2 + r^{(1)} \tag{22}$$

wherein

$$\bar{\mu}_2 = \mu_2 + h\mu_3 \tag{23}$$

and the expression $r^{(1)}$ is given by (21).

Finally, we need to approximate $u^{(3)}(0)$. The Taylor expansions for $u(3h)$, $u(2h)$ and $u(h)$ at $t=0$ allow us to achieve that

$$u_{\bar{t}tt,1} = u'''(0) + \frac{3}{2}hu^{(4)}(0) + \frac{5}{4}h^2u^{(5)}(\xi_4).$$

In according to (1) and (2),

$$u^{(4)}(0) = f(0) - a(0)\mu_2 - b(0)\mu_0.$$

Thereby we have

$$u_{\bar{t}tt,1} = \bar{\mu}_3 + r^{(2)} \tag{24}$$

where

$$\bar{\mu}_3 = \mu_3 + \frac{3h}{2}\left(f(0) - a(0)\mu_2 - b(0)\mu_0\right), \tag{25}$$

$$r^{(2)} = \frac{5}{4}h^2u^{(5)}(\xi_4). \tag{26}$$

The following difference scheme is used to approximate (1)-(2) according to (17) and (19), (22), (24).

$$y_{\bar{t}t\bar{t}t,i} + a_i y_{\bar{t}t,i} + b_i y_i - \sum_{j=0}^{i} \hbar_j \left[K_1 \left(t_i, t_j\right) D^4 y_j + K_2 \left(t_i, t_j\right) D^2 y_j + K_3 \left(t_i, t_j\right) y_j \right] = f_i,$$

$$i = 2, 3, ..., N-2, \qquad (27)$$

$$y_0 = \mu_0, \qquad (28)$$

$$y_{t,0} = \bar{\mu}_1, \qquad (29)$$

$$y_{\bar{t}t,1} = \bar{\mu}_2, \qquad (30)$$

$$y_{\bar{t}tt,1} = \bar{\mu}_3, \qquad (31)$$

where $\bar{\mu}_1$, $\bar{\mu}_2$ and $\bar{\mu}_3$ are defined as (20), (23), (25) respectively.

Error Analysis

Let u_i and y_i denote the solutions of (1) and (2), respectively. Then, the error function that is defined as $z_i = y_i - u_i$ for $0 \leq i \leq N$ will solve the following problem:

$$z_{\bar{t}\bar{t}\bar{t}\bar{t},i} + a_i z_{\bar{t}t,i} + b_i z_i - \sum_{j=0}^{i} \hbar_j [K_1(t_i,t_j)D^4 z_j + K_2(t_i,t_j)D^2 z_j + K_3(t_i,t_j)z_j] = R_i, \qquad (32)$$

$z_0 = 0,$
$z_{t,0} = r^{(0)},$
$z_{\bar{t}t,1} = r^{(1)},$
$z_{\bar{t}tt,1} = r^{(2)}.$
Here, R_i represents the truncation error as defined in (18), while $r^{(0)}$, $r^{(1)}$, and $r^{(2)}$ are specified in (20), (21), and (26), respectively.

The equation (32) can be rewritten in the following form:

$$z_{\bar{t}t\bar{t}t,i} + a_i z_{\bar{t}t,i} + b_i z_i - \sum_{j=2}^{i} \hbar_j \tilde{K}_{1,ij} z_{\bar{t}t\bar{t}t,j} - \sum_{j=1}^{i} \hbar_j \left(\tilde{K}_{2,ij} z_{\bar{t}t,j} + K_{3,ij} z_j \right) = R_i,$$
$$i = 2, 3, ..., N-2 \tag{33}$$

where
$\tilde{K}_{1,ij} = \begin{cases} K_{1,i0} + K_{1,i1} + K_{1,i2}, & j = 2, \\ K_{1,ij}, & j > 2. \end{cases}$ and $\tilde{K}_{2,ij} = \begin{cases} K_{2,i0} + K_{2,i1}, & j = 1, \\ K_{2,ij}, & j > 1. \end{cases}$

Lemma 1

The following estimate for sufficiently small values of h is provided for the error function z_i:

$$\|z\|_{\infty,\bar{\omega}_N} \leq C\left(\|R\|_{1,\omega_N} + \left|r^{(0)}\right| + \left|r^{(1)}\right| + \left|r^{(2)}\right|\right), \tag{34}$$

where C is a constant that is independent of the mesh parameter.

Proof of Lemma 1

Once we apply the discrete analogue of Gronwall's inequality, we arrive at

$$\delta_i \leq g_i + c_1 h \sum_{j=2}^{i} g_j, \quad c_1 = 3\bar{K}_1 e^{\frac{3\bar{K}_1 T}{e^{1-3\bar{K}_1 h}}}.$$

Therefore

$$|z_{\bar{t}t\bar{t}t,i}| \leq \|a\|_{\infty,\bar{\omega}_N} \left|z_{\bar{t}t,i}\right| + \|b\|_{\infty,\bar{\omega}_N} |z_i| + 2\bar{K}_2 h \sum_{j=2}^{i} \left|z_{\bar{t}t,j}\right| + \bar{K}_3 h \sum_{j=1}^{i} |z_j| + |R_i|$$

$$+\|a\|_{\infty,\bar{\omega}_N} c_1 h \sum_{j=2}^{i} \left|z_{\bar{t}t,j}\right| + \|b\|_{\infty,\bar{\omega}_N} c_1 h \sum_{j=2}^{i} |z_j| + 2\bar{K}_2 c_1 h \sum_{j=2}^{i} \left(h \sum_{k=1}^{j} \left|z_{\bar{t}t,k}\right| \right)$$

$$+\bar{K}_3 c_1 h \sum_{j=2}^{i} \left(h \sum_{k=1}^{j} |z_k| \right) + c_1 h \sum_{j=2}^{i} |R_j|$$

$$\leq \|a\|_{\infty,\bar{\omega}_N} \left|z_{\bar{t}t,i}\right| + \|b\|_{\infty,\bar{\omega}_N} |z_i| + c_2 h \sum_{j=1}^{i} \left|z_{\bar{t}t,j}\right| + c_3 h \sum_{j=1}^{i} |z_j| + |R_i| + c_1 \|R\|_{1,\omega_N} \qquad (35)$$

where

$$c_2 = 2\bar{K}_2 + \|a\|_{\infty,\bar{\omega}_N} c_1 + 2\bar{K}_2 c_1 T,$$
$$c_3 = \bar{K}_3 + \|b\|_{\infty,\bar{\omega}_N} c_1 + \bar{K}_3 c_1 T.$$

Note that the following identity holds for any mesh function v_i

$$v_{\bar{t}t,i} = v_{\bar{t}t,1} + (i-1)hv_{\bar{t}tt,1} + h^2 \sum_{k=2}^{i} (i-k) v_{\bar{t}t\bar{t}t,k},$$

from this equation, the following inequality is obtained for the error function

$$\left|z_{\bar{t}t,i}\right| \leq \left|r^{(1)}\right| + T\left|r^{(2)}\right| + Th\sum_{k=2}^{i}\left|z_{\bar{t}t\bar{t}t,k}\right|. \tag{36}$$

Using (35) and (36), we can write

$$|z_{\bar{t}t,i}| \leq q_i + \left(\|a\|_{\infty,\bar{\omega}_N} T + c_2 T^2\right) h \sum_{j=1}^{i} \left|z_{\bar{t}t,j}\right| \tag{37}$$

with

$$q_i = \left|r^{(1)}\right| + T\left|r^{(2)}\right| + T\|R\|_{1,\omega_N} + Tc_1\|R\|_{1,\omega_N} + \left(\|b\|_{\infty,\bar{\omega}_N}T + c_3T^2\right)h\sum_{j=1}^{i}\left|z_j\right|.$$

Again, using the discrete analogue of Gronwall's inequality in (37), we achieve

$$|z_{\bar{t}t,i}| \leq c_5 + (1 + c_4T)\left(\|b\|_{\infty,\bar{\omega}_N}T + c_3T^2\right)h\sum_{j=1}^{i}\left|z_j\right|,$$

here

$$c_4 = \left(\|a\|_{\infty,\bar{\omega}_N}T + c_2T^2\right)e^{\frac{\left(\|a\|_{\infty,\bar{\omega}_N}T + c_2T^2\right)T}{1-\left(\|a\|_{\infty,\bar{\omega}_N}T + c_2T^2\right)h}}$$

$$c_5 = \left|r^{(1)}\right| + T\left|r^{(2)}\right| + T\|R\|_{1,\omega_N} + Tc_1\|R\|_{1,\omega_N} + c_4T\left(\left|r^{(1)}\right| + T\left|r^{(2)}\right| + T\|R\|_{1,\omega_N} + Tc_1\|R\|_{1,\omega_N}\right).$$

Then by using the following inequality that is true for any mesh function v_i

$$|v_i| \leq |v_0| + T\left|v_{t,0}\right| + Th\sum_{k=1}^{i}\left|v_{\bar{t}t,k}\right|,$$

we get

$$|z_i| \leq T\left|r^{(0)}\right| + T^2 c_5 + (1 + c_4 T)\left(\|b\|_{\infty,\bar{\omega}_N} T + c_3 T^2\right) T^2 h \sum_{k=1}^{i} |z_k|.$$

From above inequality, applying the distinction analogue of Gronwall's inequality, it is found that

$$|z_i| \leq \alpha e^{\frac{\beta T}{1-\beta h}},$$

where

$$\alpha = |r^{(0)}|T + T^2 c_5,$$
$$\beta = (1 + c_4 T)(\|b\|_{\infty,\bar{\omega}_N} T + c_3 T^2) T^2.$$

The proof of Lemma is completed.

Theorem 2

Let $u(t)$ and y_i are the solutions of (1)-(2) and (27)-(31), respectively. Then, under the smoothness conditions $a, b, f \in C\left(\bar{\Omega}\right)$, $K_1, K_2, K_3 \in C^2\left(\bar{\Omega}^2\right)$ and $u \in C^6\left(\bar{\Omega}\right)$, for the error of the approximate solution, for sufficiently small values of h, the following estimate holds

$$\|y - u\|_{\infty,\bar{\omega}_N} \leq Ch^2.$$

Proof of Theorem 2

It is easy to demonstrate that under the smoothness conditions above

$$\|R\|_{1,\omega_N} \leq Ch^2,$$
$$\left|r^{(0)}\right| \leq Ch^2,$$
$$\left|r^{(1)}\right| \leq Ch^2,$$
$$\left|r^{(2)}\right| \leq Ch^2.$$

Together with (34), these inequalities complete the proof.

Numerical Examples

We provide numerical findings in this section that demonstrate the suggested approach. For two examples, the exact errors e_i^N for any N as follows:

$$e_i^N = |u_i - y_i|, \quad e^N = \|y - u\|_{\infty,\overline{\omega}_N}.$$

Example 1

We consider the following NVIDE:

$$u^{(4)}(t) + 8u''(t) + 16u(t) - \int_0^t \left[\left(u^{(4)}(s) + 4u''(s)\right)(t-s) + u(s)\right] ds$$

$$= \frac{16 - 3sin(2t) - (16-2t)cos(2t)}{4}, \quad 0 < t \leq 1,$$

$$u(0) = 1, \ u'(0) = 0, \ u''(0) = 0, \ u'''(0) = 0.$$

The exact solution is $u(t) = cos(2t) + tsin(2t)$. The experimental rate of convergence p^N for any N is defined

$$p^N = \frac{\ln\left(e^N/e^{2N}\right)}{\ln 2}.$$

Table 1. The numerical results for the test problem 1

t_i	u_i	y_i	e_i^{16}	y_i	e_i^{32}
0.1	0.999989	1.000000	0.000011	0.999992	0.000003
0.2	0.999837	1.000000	0.000163	0.999877	0.000040
0.3	0.999183	0.999634	0.000451	0.999295	0.000112
0.4	0.997438	0.998305	0.000867	0.997653	0.000215
0.5	0.993806	0.995206	0.001400	0.994153	0.000347
0.6	0.987303	0.988343	0.001040	0.987561	0.000258
0.7	0.976797	0.977571	0.000774	0.976989	0.000192
0.8	0.961037	0.962624	0.001587	0.961430	0.000393
0.9	0.938702	0.939758	0.001056	0.938964	0.000262
$p^{16} = 2.01$					

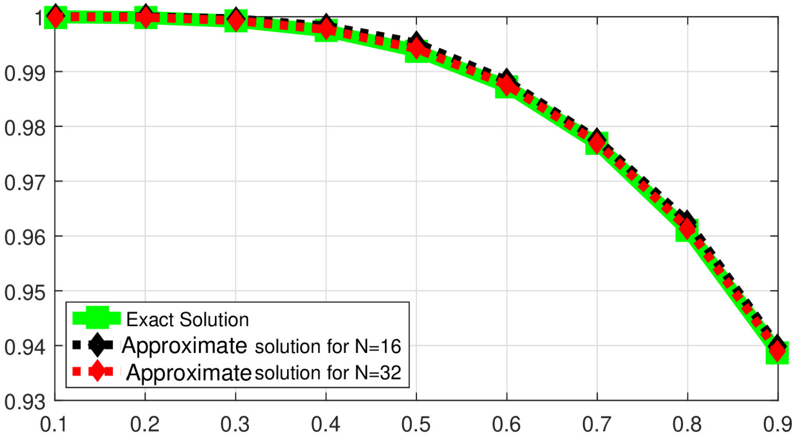

Figure 1. Numerical results of Example 1 for $N = 16, 32$

Example 2

We consider the following NVIDE:

$$u^{(4)}(t) + 2u''(t) + u(t) - \int_0^t \left[\left(u^{(4)}(s) + u''(s) \right) e^s + u(s) \right] ds$$
$$= \frac{\sin(t)\left(e^t - 3\right) + \cos(t)\left(e^t + t\right) - 1}{2}, \quad 0 < t \leq 1,$$
$$u(0) = 1,\ u'(0) = 0,\ u''(0) = 0,\ u'''(0) = 0.$$

The exact solution is $u(t) = \frac{t\sin(t)}{2} + \cos(t)$.

Table 2. The numerical results for the test problem 2.

t_i	u_i	y_i	e_i^{10}	y_i	e_i^{50}
0.1	0.999996	1.000000	0.000004	0.999997	0.000001
0.2	0.999934	1.000000	0.000066	0.999938	0.000004
0.3	0.999665	0.999850	0.000185	0.999674	0.000009
0.4	0.998945	0.999304	0.000359	0.998962	0.000017
0.5	0.997439	0.998023	0.000584	0.997466	0.000027
0.6	0.994728	0.995584	0.000856	0.994767	0.000039
0.7	0.990318	0.991490	0.001172	0.990369	0.000051
0.8	0.983649	0.985175	0.001526	0.983714	0.000065
0.9	0.974107	0.976021	0.001914	0.974186	0.000079

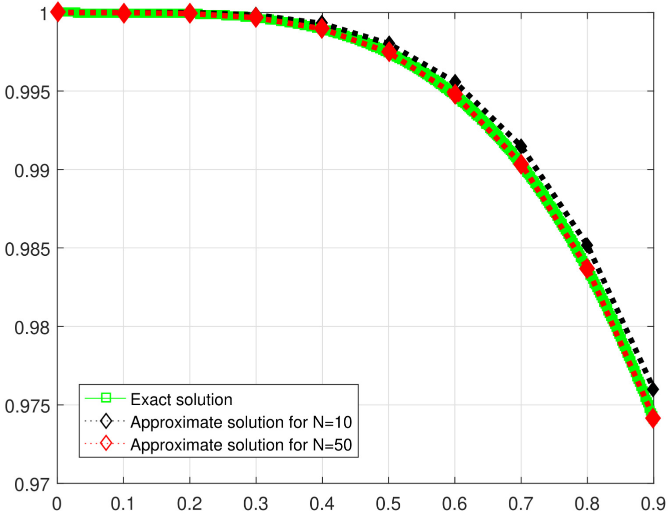

Figure 2. Numerical results of Example 2 for $N = 10,\ 50$

As clearly seen from Figures $(1)-(2)$ and Tables $(1)-(2)$, as the value of N increases, the approximate solution converges to the exact solution, which confirms the validity of our theoretical analysis.

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